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Critical dynamics and the potential moving approximation

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Abstract. The real-space time-dependent renormalisation group approach is applied to the study of a square lattice. A decimation and a spin-block transformation are performed according to the potential moving trick suggested by Kadanoff. A value $z = 2{\cdot}2$ is found for the dynamic index characterising the two-dimensional lattice.

1. Introduction

Systems close to a critical point exhibit anomalies in static and dynamic properties. These anomalies have been the subject of many different studies. One of the leading methods is the renormalisation group (RG) approach, which provides a verification of the hypothesis of universality, a way to extract the critical exponent, and a method to calculate thermodynamic and correlation functions. Most of the dynamical models are based on the Glauber model (Glauber 1963) and were analysed using the ϵ-expansion around four and six dimensions (see, for example, Hohenberg and Halperin 1977). Recently the real-space RG calculations have been applied to the study of dynamic phenomena (Achiam 1978a, b, 1979, Achiam and Kosterlitz 1978, Kinzel 1978, Mazenko *et al* 1978, Suzuki *et al* 1979, Chui *et al* 1979). The values obtained for the critical exponent, z, which characterised the dependence of the relaxation time on the correlation length, seem to be in good agreement with the results obtained using more conventional methods such as Monte Carlo simulations (Stoll *et al* 1973) and high-temperature series (Yahata and Suzuki 1969, Rácz and Collins 1976). If the results concerning the dynamics are as accurate as the results for the statics (and we hope so), the real-space time-dependent renormalisation group approach (TRG) can be a powerful tool which provides a numerical estimate as well as a better understanding of the critical dynamic scaling and the critical modes.

In this paper we are studying the implication of Kadanoff's 'potential moving approximation' (Kadanoff 1976) on the TRG. This approximation has the advantage of being simple and easy to interpret, gives reasonable static critical exponents, and includes the Migdal approximation (Migdal 1975) as a limit. The potential moving approximation is easy to extend to any lattice dimensionality and scale factor b. However, as the dimensionality d diverges from 1D, the critical static exponent becomes worse.

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We obtain in 2D, $z = 2{\cdot}2$. This value falls within the range of other estimates obtained by using other methods. It is surprisingly close to the value we obtained previously using the TRG and the second-order cumulant approximation on a triangular lattice (Achiam 1978b). The study of the triangular lattice with the cumulant approximation has the advantage of fairly good static exponents; hence this value of z is quite reliable.

This paper is organised as follows. In § 2 we review the basic ideas of the RG technique and the potential moving approximation. These ideas and notations are used in the TRG and are reported in more detail in the article by Kadanoff (1976). The general aspects of the TRG are discussed in § 3. They are used in § 4 to calculate z using the potential moving approximation with the decimation and spin-block transformations. The technique and the results are discussed in § 5, and compared to other approximations which are based on the Migdal approximation.

2. The 'potential moving approximation'

In the application of the RG to the equilibrium state of an Ising spin system, one starts with a probability distribution, $P(\sigma)$, which is a functional of all the spins $\{\sigma_i\}$, $\sigma_i = \pm 1$, defined on a lattice. Then one defines a transformation of $P(\sigma)$ to a new probability distribution of a set of spin variables $\{\mu_i\}$, $\mu_i = \pm 1$. The $\{\mu\}$ are on a lattice which has the same symmetry as the $\{\sigma\}$ lattice, but has a lattice constant larger by a factor of b. The transformation is of the form

$$P'(\mu) = \sum_{\{\sigma\}} T(\sigma, \mu) P(\sigma) \tag{2.1}$$

where T is subject to the following conditions.

(i) It has to preserve the normalisation of P:

$$\sum_{\{\mu\}} T(\sigma, \mu) = 1. \tag{2.2}$$

(ii) It has to be non-negative:

$$T(\sigma, \mu) \geqslant 0. \tag{2.3}$$

(iii) It should not change the symmetry of the lattice.

The probability distribution P can be represented by a set of interactions, $\boldsymbol{K} = \{K_a\}$, which are coupled to spin operators, $\boldsymbol{S}(\sigma) = \{S_a(\sigma)\}$,

$$P(\sigma) = \exp\left[\sum_a K_a S_a(\sigma)\right] / Z \equiv \exp\{\bar{H}\}/Z \tag{2.4}$$

where the partition function Z normalises P.

According to equations (2.1)–(2.3), $P'(\mu)$ can be written in a form similar to (2.4), using a renormalised reduced Hamiltonian,

$$\bar{H}'(\mu) = \sum_a K'_a S_a(\mu) \tag{2.5}$$

and the RG transformation can be considered to be a transformation upon the parameter space $\boldsymbol{K}$,

$$\boldsymbol{K}' = \mathbf{R}\boldsymbol{K}. \tag{2.6}$$

The fixed point of this transformation, $\boldsymbol{K}^* = \mathbf{R}\boldsymbol{K}^*$, is associated with a critical point (or a zero correlation) (Wilson and Kogut 1974). The eigenvalues of $\mathbf{R}$ are associated with the critical static exponent.

Usually, the transformation (2.1) can be performed only approximately. In one of the approximations, the so-called 'variational method', one calculates $\bar{H}'_A$,

$$\exp[\bar{H}'_A(\mu)] = \mathrm{Tr}_\sigma T(\sigma, \mu) \exp[\bar{H}(\sigma) - \Delta(\mu, \sigma)] \tag{2.7}$$

such that

$$\mathrm{Tr}_\mu \mathrm{Tr}_\sigma T(\sigma, \mu) \exp[\bar{H}(\sigma)]\Delta(\mu, \sigma) = 0. \tag{2.8}$$

The new reduced Hamiltonian will generate a free energy which is smaller than the exact one. The error in the calculation will be of second order in Δ.

Kadanoff (1976) suggested a Δ of the form

$$\Delta(\mu, \sigma) = \sum_i q_i a_i(\sigma) \tag{2.9}$$

where $a_i(\sigma)$ are local interactions which appeared in $\boldsymbol{S}(\sigma)$ (e.g. nearest-neighbour coupling in one direction), and

$$\sum_i q_i = 0. \tag{2.10}$$

Under conditions (2.9)–(2.10) the quantity $\bar{H}(\sigma) - \Delta(\mu, \sigma)$ can be interpreted as a Hamiltonian from which part of the interactions is moved from one point on the lattice to another point such that the total amount of interactions (bonds) is not changed.

Two examples of RG transformation, both leading to the same recursion relations, have been discussed by Kadanoff. We will apply them to the dynamical models; hence we will describe them briefly. Starting with a square lattice, one divides the spins in the x direction into blocks of length of b lattice constants. The interactions in the y direction are moved away via the potential moving trick to the edges of the blocks, in the decimation transformation (figure 1*b*), or all to one special location on the lattice, in the spin-block transformation (figure 1*c*). Now the blocks in the x direction are decoupled and the RG transformation in this direction can be performed according to one of the two T:

$$T_d = \prod_{k,y} (\mu_{kbL,y} - \sigma_{kbL,y}) \tag{2.11}$$

where L is the lattice constant, in the decimation transformation, and

$$T_b = \prod_{k,y} (\cosh K/2)[1 + (\sigma_{kbL,y} + \sigma_{kbL+1,y})\mu_{kbL,y} \tanh(p) + \sigma_{kbL,y}\sigma_{kbL+1,y} \tanh^2 p] \exp(-K\sigma_{kbL,y}\sigma_{kbL+1,y}) \tag{2.12}$$

in the block-spin transformation. In equations (2.11) and (2.12) we assumed a Hamiltonian with nearest-neighbour interactions only:

$$\bar{H} = K \sum_{\langle i,j \rangle} \sigma_i \sigma_j \tag{2.13}$$

where i, j are nearest neighbours. The parameter p in (2.12) is fixed by

$$\tanh^2 p = \tanh K \tag{2.14}$$

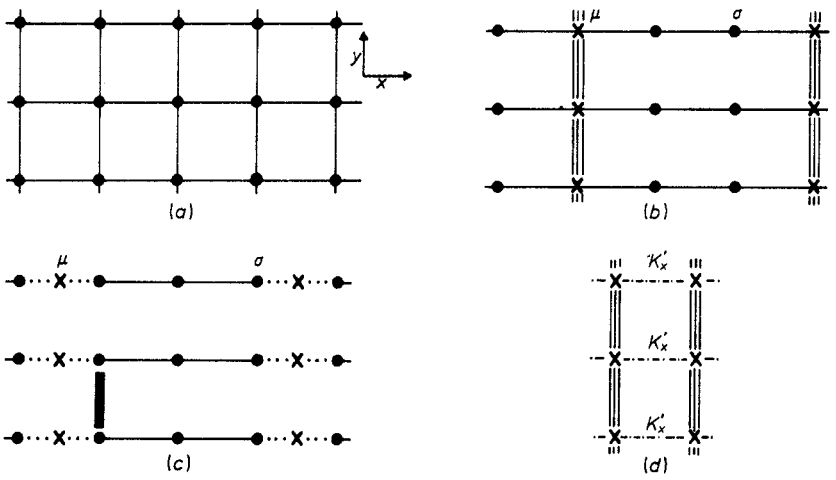

Figure 1. (*a*) The lattice before the RG transformation. (*b*) The first potential moving in the decimation transformation. The *x* denotes the μ variables which are not summed up. (*c*) The first potential moving in the spin-block transformation. The full lines and the dotted lines represent *K* and *p* interactions, respectively. The *x* denote the μ variables. All the old *y* interactions are collected at the broad line. (*d*) The effective interactions after the RG transformation in the *x* direction.

so that condition (2.2) is fulfilled. In both transformations the result is

$$K'_x = \tan^{-1}(\tanh^b K_x) \equiv R^b K_x. \tag{2.15}$$

In the decimation transformation one can perform once again the potential moving trick, now in the *x* direction. Hence

$$K_x(b) = bR^b K_x. \tag{2.16}$$

After the previous *x* transformation the *y* interactions were

$$K'_y = bK_y \tag{2.17}$$

and after the RG in the *y* direction they become

$$K_y(b) = R^b(bK_y). \tag{2.18}$$

The spin-block transformation is a little more complicated. Before performing the RG in the *x* direction, infinitesimally small interactions are removed from each σ–μ coupling and are put into the σ–μ coupling of the two spins which are still coupled in the *y* direction. This new coupling becomes very strong and hence we shift the *y* coupling from the σ to the μ. We can now perform the *x* transformation, leading to (2.15). The *y* interaction is spread over all the new lattice, leading to (2.17) (see figure 1*d*). This procedure is repeated in the *y* direction, leading finally to the relations (2.15) and (2.18).

We would like to stress three points before we continue to the description of the TRG.

(a) We limit our discussion to two dimensions. The generalisation to higher dimensionality is straightforward (Kadanoff 1976).

(b) In the Migdal transformation the recursion relation (2.15) is also used for the other directions.

(c) Within the potential moving approximation we are allowed to change a finite number of interactions, and to assume that the other interactions of the same type are not changed.

3. The Glauber model and the time-dependent renormalisation group transformation

The dynamics we are going to study is that of the Glauber–Ising model (Glauber 1963). This model describes the time-dependent behaviour of a large interacting spin system whose equilibrium is determined by an Ising Hamiltonian. The system is brought to a constrained equilibrium state. Then, at time $t=0$ the constraint is removed and the system relaxes towards the final equilibrium via an interaction with a heat bath. The heat bath is not treated explicitly in the model, and during the relaxation neither the magnetisation nor the energy of the system is conserved. Only one spin is allowed to flip at a time, with transition probability rate $W_i(\{\sigma\})$. This procedure can be described by an empirical master equation for the spin probability distribution, $P(\{\sigma\}; t)$, and a bare time scale, τ, of a spin system, $\{\sigma = \pm 1\}$,

$$\begin{aligned} \mathrm{d}P(\{\sigma\}; t)/\mathrm{d}t &= -\sum_j W_j(\sigma_1, \ldots, \sigma_j, \ldots, \sigma_n)P(\sigma_1, \ldots, \sigma_j, \ldots, \sigma_n; t) \\ &\quad + \sum_j W_j(\sigma_1, \ldots, -\sigma_j, \ldots, \sigma_n)P(\sigma_1, \ldots, -\sigma_j, \ldots, \sigma_n; t) \\ &\equiv L(\sigma)P(\sigma, t) \equiv -\sum_j (1-p_j)W_j(\sigma)P(\sigma, t) \end{aligned} \tag{3.1}$$

where p_j is a spin-flip operator: $p_j f(\sigma_1, \ldots, \sigma_j, \ldots, \sigma_n) = f(\sigma_1, \ldots, -\sigma_j, \ldots, \sigma_n)$, and the transition probability satisfies the detailed balance which ensures the ergodicity of the system:

$$(1-p_j)W_j(\sigma)P_e(\sigma) = 0 \tag{3.2}$$

where e denotes equilibrium. The master equation (3.1) can be written in a slightly different form:

$$\mathrm{d}P(\sigma, t)/\mathrm{d}t = -\mathscr{L}\phi(\sigma, t) \tag{3.3}$$

where $\phi(\sigma, t)$ measures the deviation from equilibrium:

$$\phi(\sigma, t) = P(\sigma, t)/P_e(\sigma) \tag{3.4}$$

and $P_e(\sigma) = P(\sigma, \infty)$ is of the form (2.4). In the following we will restrict ourselves to $\bar{H}$ with nearest-neighbour interactions (2.13) only. The operator $\mathscr{L}$ is obtained from (3.1)–(3.2),

$$\mathscr{L} \equiv \sum_i \mathscr{L}_i, \qquad \mathscr{L}_i = P_e W_i(1-p_i). \tag{3.5}$$

For further properties of the Liouville operator L (or $\mathscr{L}$), the reader is referred to Yahata and Suzuki (1969) and references therein. We shall only note that, since W_i does not depend on the history of the system, the Glauber model is a Markovian one.

The relation (3.2) does not determine W_i uniquely. We will use (Achiam and Kosterlitz 1978)

$$W_i(\sigma_i) = [P_e(\sigma_1, \ldots, -\sigma_i, \ldots, \sigma_n)/P_e(\sigma_1, \ldots, \sigma_i, \ldots, \sigma_n)]^{1/2}. \tag{3.6}$$

With the present Hamiltonian, (2.13), W_i is

$$W_i(\sigma_i) = \exp\left(-K\sigma_i \sum_j \sigma_j\right) \tag{3.7}$$

(the j are nearest neighbours of i).

The TRG is the standard RG transformation, (2.7), applied to the master equation (3.1) or (3.3), followed by time scaling $\tau' = b^z\tau$ which leaves the master equation invariant. More explicitly, expanding ϕ as follows,

$$\phi = 1 + \sum_a h_a(t)O_a(\sigma) \equiv 1 + (\boldsymbol{h} \cdot \boldsymbol{O}), \tag{3.8}$$

the master equation reads:

$$P_e(\sigma)\mathrm{d}/\mathrm{d}t[\boldsymbol{h} \cdot \boldsymbol{O}(\sigma)] = -\mathscr{L}[\boldsymbol{h} \cdot \boldsymbol{O}(\sigma)]. \tag{3.9}$$

Under the RG transformation (2.7), equation (3.9) becomes

$$P'_e(\mu)\,\mathrm{d}/\mathrm{d}t\,[\boldsymbol{h}' \cdot \boldsymbol{O}(\mu)] = -\mathscr{L}'[\boldsymbol{\Omega h} \cdot \boldsymbol{O}(\mu)] \tag{3.10}$$

where $\boldsymbol{h}' = \boldsymbol{\Lambda h}$ is the static RG transformation of the parameter $\boldsymbol{h}$, $\mathscr{L}'$ is a function of $P'(\mu)$ as determined by (3.6), and the matrix $\boldsymbol{\Omega}$ is defined by the transformation of the RHS of (3.9). In the present case the operator

$$\boldsymbol{O} = O_1(\sigma) = \sum_i \sigma_i \tag{3.11}$$

forms an invariant subspace. Hence both $\boldsymbol{\Lambda}$ and $\boldsymbol{\Omega}$ are reduced to two scalars λ, ω. The scale factor b^z which is needed to restore the initial form of the master equation is just the ratio of these two scalars,

$$b^z = \lambda/\omega. \tag{3.12}$$

4. The transformation of the master equation

In this section we perform the RG transformation of equation (3.9) into (3.10). The LHS of equation (3.10) is the static RG transformation of the magnetic field. In the Kadanoff approximation, the decimation with $b = 2$ gives

$$\lambda_d = \{1 + 2\tanh(K_x^*)[1 + \tanh^2(K_x^*)]^{-1}\}^d \tag{4.1}$$

while the spin-block transformation gives

$$\lambda_b = [2\tanh^{1/2}K_x^*(1 + \tanh K_x^*)(1 + \tanh^2 K_x^*)^{-1}]^d. \tag{4.2}$$

In order to perform the transformation of the RHS of (3.9), we will focus our attention on a typical term, $W_iP_e\sigma_i$, in this expression. Following equations (3.5) and (3.11) the RHS of (3.9) is

$$\mathscr{L}(\boldsymbol{h} \cdot \boldsymbol{O}(\sigma)) = 2\sum_i W_iP_e\sigma_i. \tag{4.3}$$

The combination $W_i P_e$ is P_e from which the contribution of the bonds around σ_i is omitted (Achiam and Kosterlitz 1978) (figure 2).

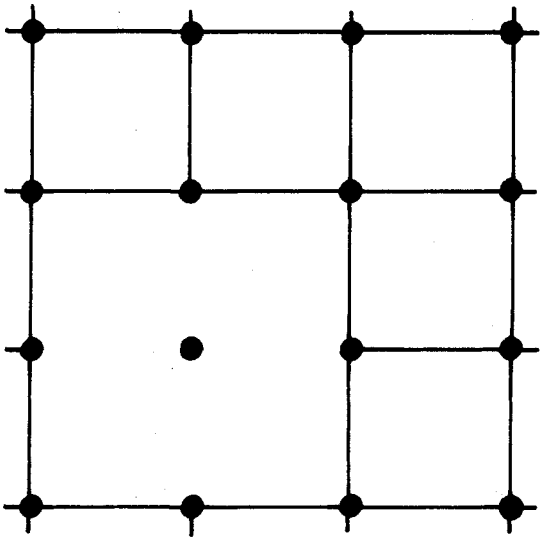

Figure 2. The interactions which appear in $W_i P_e(\sigma)$ are the ones usually in $P_e(\sigma)$ except for those around σ_i.

The decoupling of σ_i from its neighbours is an essential feature of $W_i P_e$, and is the result of the detailed balance condition. Thus the trace of σ_i in the ith term can be done exactly in the RG transformation:

$$W_i P_e \operatorname{Tr}_{\sigma_i} \sigma_i T(\mu, \sigma). \tag{4.4}$$

Generally, the trace of the rest of the σ cannot be performed so simply. We use the potential moving trick in order to decouple different directions and to have an effective one-dimensional problem where the trace can be performed. However, if we move exactly the same potential as in the static case we create a new coupling to σ_i in some directions, with a negative interaction. To solve this problem we perform another potential moving trick and spread these negative interactions over all other interactions in this direction. Since there is a finite number of extra negative bonds and an infinite number of other bonds in the same direction, this potential moving trick will not change the magnitude of any interaction except for the extra negative interaction which becomes null.

This extra approximation is still within the error of the approximation (2.7), as was discussed at the end of § 2. We have to repeat it after each standard potential move, so that the σ_i (or the corresponding transformed spin) will be decoupled in all directions, as demanded by (4.4).

In the following we will demonstrate this procedure in the relatively simple situation where σ_i is an intermediate spin (see below). The other situations are discussed in the appendices.

Suppose that σ_i is an intermediate spin, i.e. it does not appear in (2.11) (or 2.12) (see figure 3 (a, b)). This spin is decoupled from its neighbours in the x direction due to W_i.

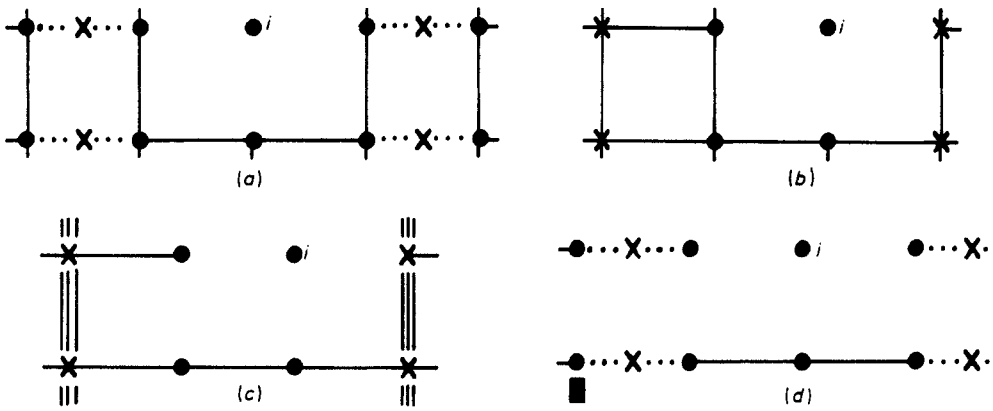

Figure 3. (*a*) The interactions in $W_i(\sigma)P_e(\sigma)$ for a typical intermediate spin σ_i. The RG is a decimation transformation. (*b*) As in (*a*) except for a spin-block transformation instead of a decimation transformation. (*c*) The configuration of the interactions after performing the potential moving trick and the cancellation of the negative bonds. (*d*) As in (*c*) except for a spin-block transformation instead of a decimation transformation.

However, after the potential moving trick it will be coupled in the y direction with a negative interaction. This artificial negative interaction is the result of the Δ in the approximation (2.7). It can be eliminated by extracting an infinitesimal amount of interaction from all the other y bonds. The sum of the infinitesimals suffices to annul the two negative y bonds, but of course each of the other y bonds is not affected. Now the σ_i is decoupled (figure 3*c*, *d*) and the $\mathrm{Tr}_{\sigma_i}\sigma_i$ will contribute zero.

The calculation of the contribution from $\mathscr{L}_i\sigma_i$ where σ_i is not an intermediate spin is more complicated. Since σ_i is not intermediate, it is connected to some μ which will be denoted as μ_i. We will call the space around μ_i which is bound by its four next-nearest neighbours 'the dynamic region'. $\mathscr{L}_i(\sigma)$ differs from $P_e(\sigma)$ only in the dynamic region. Thus the summation over the σ_j, where j is out of the dynamic region, is *exactly* the same summation as in the static RG transformation of $P_e(\sigma)$. In other words, the summation over $\{\sigma_j\}$ results in $P'_e(\mu)$ when the contribution from the dynamic region is omitted. That is, we eliminated the dynamic region interaction from $P'_e(\mu)$ by multiplying with $W'_i(\mu)$ and dividing by the contribution, A, to the free energy which results from the RG transformation in the dynamic region, i.e.

$$P'_e(\mu)W'_i(\mu)/A. \tag{4.5}$$

We still have to calculate the contribution from the summation over the dynamic region spins σ_i. This sum is performed explicitly in the appendices for both the decimation and spin-block transformations. It is shown that this sum is proportional to μ_i,

$$d\mu_i. \tag{4.6}$$

After summing over i in (3.5), using (4.5) and (4.6) we obtain:

$$h\frac{cd}{A}\sum_i P'_e(\mu)W'_i(\mu)\mu_i = \omega h \sum \mathscr{L}_i(\mu)\mu_i$$

$$= \omega\mathscr{L}'(\mu)hO(\mu) \tag{4.7}$$

where c is a combinatorial factor.

The master equation after the RG transformation is

$$\tau \frac{\mathrm{d}}{\mathrm{d}t} P'_e \phi' = \omega/\lambda \, \mathscr{L}' \phi' \tag{4.8}$$

and the time scaling $\tau' = \tau\lambda/\omega$ will keep the master equation in an invariant form.

The ω are calculated in the appendices. The results are as follows.

In the decimation transformation,

$$\omega_d = \frac{\cosh^2[K_x(b)b^{1-d}]\cosh^2[K_y(b)b^{2-d}]\ldots\cosh^2[K_l(b)b^{l-d}]}{\cosh^2[K_x]\cosh^2[bK_y]\ldots\cosh^2[b^{l-1}K_l]}$$

$$\xrightarrow[T\to T_c]{} [\cosh^2(K_x^* b^{1-d})/\cosh^2(K_x^*)]^d \tag{4.9}$$

where we have a d-dimensional hypercubic lattice with $l = 2, 3 \ldots$, for $y, z, \ldots$, correspondingly. The K are defined in (2.15)–(2.18).

In the spin-block transformation:

$$\omega_b = (2a_x)(2a_y)\ldots(2a_d)\omega_d \tag{4.10}$$

where

$$a_l = \tanh^{1/2}(b^{l-1}K_l)[1+\tanh(b^{l-1}K_l)]^{-1} \to \tanh^{1/2}(K_x^*)[1+\tanh(K_x^*)]^{-1}.$$

5. Discussion

The results obtained in § 4 can be tested in one dimension. In this limit the transformations become exact, and can be compared with the exact result of Glauber (1963). The fixed point in 1D is

$$\tanh K^* = 1. \tag{5.1}$$

The factor $2a_x$ is one; hence $\omega = \omega_d = \omega_b$. For $b = 2$ we get

$$\omega = \tfrac{1}{2}, \qquad \lambda = 2. \tag{5.2}$$

Thus, according to (3.12),

$$z = 2 \tag{5.3}$$

which is the exact result.

Another interesting limit is the Migdal (1975) approximation. This approximation can be obtained as a limit of the Kadanoff approximation if we assume that all the results which were obtained in the RG transformation in the x direction are valid also in the other directions. In our results this is implied by equations (4.5)–(4.6). Hence, for the decimation ($b = 2$),

$$\omega_d = [\cosh^2(K^*/2)/\cosh^2(K^*)]^d. \tag{5.4}$$

For $d = 2$, $K^* = 0{\cdot}609$. Hence

$$z = 2{\cdot}5. \tag{5.5}$$

This result differs from the one obtained by Chui *et al* (1979). They started with the

original factorisation–decimation scheme of Migdal (1975), and assumed the equation of motion for *one* block. This assumption leads to $\omega_d = [\cosh^2 K(b)/\cosh^2 K]^d \to 1$ at the fixed point. Another problem which has arisen in their calculation is common to all decimation calculations. We used (4.1) in order to derive (5.5). However, (4.1), which was obtained by applying the RG transformation on a small magnetic field, is inconsistent with the original decimation idea:

$$\beta/\nu = 0. \tag{5.6}$$

β, ν are the static indices characterising the phase diagram and correlation length. Equation (5.6) results from the fact that in the decimation transformation the spin's scale factor (Wilson and Kogut 1974), $b^{-\beta/\nu}$, is unity. To overcome this contradiction, we have to use the spin-block transformation. Although this transformation has defects (Kadanoff 1976), which will not be discussed here, it does not suffer from this inconsistency. Using (4.6), (4.2) and (5.4) we get, in the spin-block approximation,

$$z = 2{\cdot}24. \tag{5.7}$$

Now it is time to compare our result (5.7) to other estimates of z which have appeared in the literature. We have already discussed the result obtained by Chui *et al* (1979) and its connection to the potential moving approximation. Suzuki *et al* (1979) also tried to calculate z using the potential moving trick. They first moved all the bonds within a block to its edge, and then calculated the equation of motion of the first moment of the probability distribution after a decimation along the edges (Achiam 1978a). The static exponents and fixed point are not the ones appearing in the potential moving approximation. Their calculation suffers from the fact that they were based on decimation in two dimensions, in addition to other uncontrolled approximations.

Other values of z are: real-space RG transformation to the second order of the cumulant expansion on a triangular lattice, $z = 2{\cdot}2$ (Achiam 1978b); high-temperature series, $z = 2{\cdot}125$ (Rácz and Collins 1976), $z = 2{\cdot}0$ (Yahata and Suzuki 1969); and Monte Carlo simulation, $z = 1{\cdot}85$ (Stoll *et al* 1973).

Appendix 1. The decimation transformation of equation (4.3)

First we perform the infinitesimal shifting of all the y bonds so that the actual $\mu_i(\mu_2$ in figure A1) is decoupled from its neighbour. Then the summation in the x direction is straightforward. All the bonds in the x direction which are not between the actual $\mu_i(\mu_2$

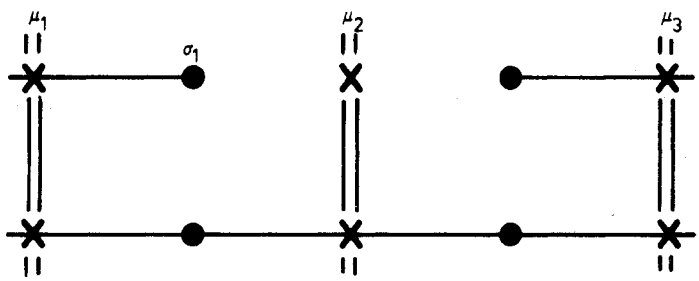

Figure A1. The decimation transformation ($b = 2$) of the master equation—the bond configurations after the potential moving trick is applied to the y bonds. Each line represents an interaction of K strength.

in figure A1) and its nearest-neighbour μ (μ_1 and μ_3 in figure A1) give the usual static contribution to $P'_e(\mu)$,

$$\tilde{P}_e W_i \cosh^2 K'_x (2^{2b-2} \cosh^{2b} K_x)^{-1} \tag{A1.1}$$

where the last factor came from the contribution to the free energy. The $\sim$ means that we have not integrated yet in the y direction. The bonds between μ_1 and μ_3 contribute

$$\mu_2 \operatorname{Tr}_{\sigma_1\sigma_2} \exp[K(\sigma_1\mu_1 + \sigma_2\mu_2)] = \mu_2 4 \cosh^2 K$$

or for general b

$$\mu_2 2^{2b-1} \cosh^{2b-2} K. \tag{A1.2}$$

From equations (A1.1) and (A1.2) it follows that the total contribution to (3.10) is

$$2 \frac{\cosh^2[K_x(b)/b]}{\cosh^2 K_x} \tilde{\sum_i} \tilde{\mu}_i W_i \tilde{P}_e(\mu) \tag{A1.3}$$

where $\tilde{\mu}_i$ means that spins which still have to be integrated in the y direction are also included. $K_x(b)$ is given by equation (2.16).

The summation in the y direction is done exactly as in the x direction after a potential moving trick of the x bonds, leading to

$$2\cosh^2[K_y(b)] \cosh^2[K_x(b)/b][\cosh^2(K'_y b) \cosh^2 K_x]^{-1} \sum_i \mu_i W_i P'_e \tag{A1.4}$$

where all the different K were defined in equations (2.15)–(2.18). Equations (A1.4) and (4.3) give

$$\omega_d = \cosh^2 K_y(b) \cosh^2 K'_x [\cosh^2 K'_y \cosh^2 K_x]^{-1}. \tag{A1.5}$$

For a d-dimensional system, equation (A1.5) becomes (4.5).

Appendix 2. The spin-block transformation of equation (4.3)

This procedure is quite similar to the decimation transformation. After performing the first potential moving trick, such that all the y bonds were put between two points of the lattice, the situation is as follows. The particular σ_i of the ith term in (4.3) (i.e. σ_3 in figure A2) is coupled with negative interaction to its neighbours in the y direction. These two bonds can be cancelled out by another potential moving trick which changes the other y bonds by a negligible magnitude only. After the shifting the $\sigma_i(\sigma_3)$ is

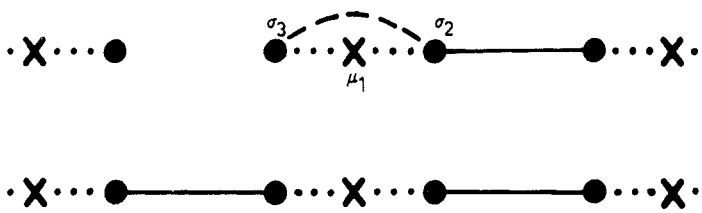

Figure A2. The spin-block transformation of the master equation—the bond configurations in the x direction after applying the potential moving trick on the y bonds. The full lines, dotted lines and broken lines represent K, p, $-K$ interactions, respectively.

coupled to its neighbours in the other block (σ_2) and to the nearest μ (μ_1). The trace over σ_i can be performed easily, leading to

$$\begin{aligned} \mathrm{Tr}_{\sigma_i}\sigma_i T_b &= \mu \tanh^{1/2} K_x (1+\tanh K_x)^{-1} \\ &\equiv \mu a_x \end{aligned} \tag{A2.1}$$

where (2.14) has been used in the derivation of (A2.1).

The trace over the other spins in the x direction which are in the two blocks of size b near σ_i gives

$$\mu a 2^{2b-2} \cosh^{2b-2}(K_x) 2.$$

The last factor of 2 is the result of two configurations that lead to the same contribution.

Now we spread the y bonds over the lattice which was scaled in the x direction. We take care not to put the y bonds around the μ_i. If this μ_i is an intermediate spin in the y direction, the trace will be zero.

From the two μ_j near the edge of the block we get a contribution

$$2^{2b-2} \cosh^{2b-2}(bK_y) 2. \tag{A2.2}$$

This procedure is repeated in all d dimensions. We obtain a new scaled lattice in which all the potentials are those of $P'_e(\mu)$ except that the bonds around the particular μ_i are missing. Taking into account the contribution to the free energy, we finally obtain equation (4.6).

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